C^*-algebras of separated graphs

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P. Ara, *Purely infinite simple reduced graph C*-algebras of one-relator graphs*, arXiv:1203.2815v1 [math.OA].

Some motivation: the C^*-algebras $U^{\mathrm{nc}}_{m,n}$

Larry Brown (1981) studied the C^*-algebra U^{nc}_n. Concretely the algebra U^{nc}_n was defined as the universal algebra generated by elements u_{ij}, $1 \leq i, j \leq n$, subject to the relations making (u_{ij}) a unitary $n \times n$ matrix.

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McClanahan raised the problem of computing the K-theory of $U^{\mathrm{nc}}_{m,n}$, establishing a conjecture and proving it in several cases.

Definition

A directed graph is given by $E = (E^0, E^1, s, r)$, where E^0 and E^1 denote the sets of vertices and edges of E, respectively, and $s, r : E^1 \to E^0$ are the source and range maps.

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Definition

A path in E is $\alpha = e_1 e_2 \cdots e_n$ where $e_i \in E^1$ and $r(e_i) = s(e_{i+1})$ for $i < n$. The *length* of such a path is $|\alpha| := n$. Paths of length 0 are identified with the vertices of E.

Separated graphs

Definition

A *separated graph* is a pair (E, C) where E is a graph, $C = \bigsqcup_{v \in E^0} C_v$, and C_v is a partition of $s^{-1}(v)$ (into pairwise disjoint nonempty subsets) for every vertex v:

$$s^{-1}(v) = \bigsqcup_{X \in C_v} X.$$

(In case v is a sink, we take C_v to be the empty family of subsets of $s^{-1}(v)$.)
The constructions we introduce revert to existing ones in case $C_v = \{s^{-1}(v)\}$ for each $v \in E^0$. We refer to a *non-separated graph* in that situation.

Definition

For any separated graph (E, C), the full graph C*-algebra of the separated graph (E, C) is the universal C*-algebra with generators $\{v, e \mid v \in E^0,\ e \in E^1\}$, subject to the following relations:

(V) $vw = \delta_{v,w} v$ and $v = v^*$ for all $v, w \in E^0$,

(E) $s(e)e = er(e) = e$ for all $e \in E^1$,

(SCK1) $e^* f = \delta_{e,f} r(e)$ for all $e, f \in X$, $X \in C$, and

(SCK2) $v = \sum_{e \in X} ee^*$ for every finite set $X \in C_v$, $v \in E^0$.

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In case (E, C) is trivially separated, $C^*(E, C)$ is just the classical graph C*-algebra $C^*(E)$.

Example

Assume that (E, C) is a separated graph and that $|E^0| = 1$. Then we have

$$C^*(E, C) \cong *_{X \in C}\, \mathcal{O}_{|X|},$$

that is, $C^*(E, C)$ is a free product over $\mathbb{C}$ of Cuntz algebras of type $(1, |X|)$, for $X \in C$.

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Note that if $|X| = 1$ for all $X \in C$ then

$$C^*(E, C) \cong C^*(\mathbb{F}_r),$$

a full group C*-algebra of the free group $\mathbb{F}_r$ on $r = |E^1|$ generators.

Example

Let $1 \leq m \leq n$. Let us consider the separated graph $(E(m,n), C(m,n))$, where $E(m,n)$ is the graph consisting of two vertices v, w and with

$$E(m,n)^1 = \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\},$$

with $s(\alpha_i) = s(\beta_j) = v$ and $r(\alpha_i) = r(\beta_j) = w$ for all i, j, and $C(m,n)$ consists of two elements $X = \{\alpha_1, \dots, \alpha_n\}$ and $Y = \{\beta_1, \dots, \beta_m\}$.

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Write

$$A_{m,n} := C^*(E(m,n), C(m,n)).$$

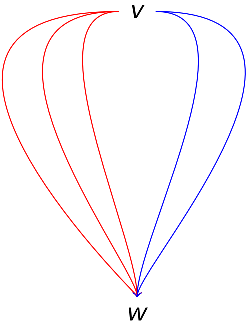

Figure: The separated graph $(E(2,3), C(2,3))$

Lemma

There is a natural isomorphism

$$\gamma \colon U^{\mathrm{nc}}_{m,n} \to wA_{m,n}w$$

given by

$$\gamma(X_{ji}) = \beta_j^* \alpha_i, \quad \gamma(X_{ji}^*) = \alpha_i^* \beta_j$$

Note that

$$\gamma(\sum_{i=1}^{n} X_{ji}X_{ki}^*) = \sum_{i=1}^{n} \beta_j^* \alpha_i \alpha_i^* \beta_k = \beta_j^* \beta_k = \delta_{jk} w$$

and similarly $\gamma(\sum_{j=1}^{m} X_{ji}^* X_{jk}) = \delta_{ik} w$ so γ is a well-defined homomorphism, which is shown to be an isomorphism.

Since $v \sim n \cdot w \sim m \cdot w$, we get from the above

$$A_{m,n} \cong M_{n+1}(wA_{m,n}w) \cong M_{n+1}(U^{\mathrm{nc}}_{m,n}) \cong M_{m+1}(U^{\mathrm{nc}}_{m,n}).$$

Amalgamated free products

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Given $\iota \colon A_0 \to A_\iota$, $\iota \in I$, the **amalgamated free product** $A = *_{A_0} A_\iota$ is a C^*-algebra A, together with $\rho_\iota \colon A_\iota \to A$ such that

$$\rho_\iota \circ \iota = \rho_{\iota'} \circ \iota' \colon A_0 \to A$$

for all ι, ι', and such that given any other $*$-homomorphisms $\delta_\iota \colon A_\iota \to B$ with $\delta_\iota \circ \iota = \delta_{\iota'} \circ \iota'$ for all ι, ι' there is a unique $\delta \colon A \to B$ such that $\delta \circ \rho_\iota = \delta_\iota$.

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We have:

$$C^*(E, C) \cong *_{A_0} A_X,$$

the amalgamated free product of A_X over A_0.

Definition

Let $B \subseteq A$ be an inclusion of C^*-algebras. A **conditional expectation** is a map $\Phi \colon A \to B$ such that

1. $\Phi(b) = b$ for all $b \in B$.
2. $\Phi(ab) = \Phi(a)b$ and $\Phi(ba) = b\Phi(a)$ for all $a \in A$ and $b \in B$.
3. Φ is positive: $\Phi(x) \geq 0$ if $x \geq 0$.
4. $\|\Phi(a)\| \leq \|a\|$ for all $a \in A$.

The conditional expectation $\Phi \colon A \to B$ is *faithful* if, for $x \geq 0$, $\Phi(x) = 0$ implies $x = 0$.

A canonical conditional expectation $\Phi\colon C^*(E) \to C_0(E^0)$ can be defined, satisfying:

$$\Phi(\gamma e f^* \nu^*) = 0, \qquad e \neq f\,,$$

$$\Phi(\gamma e e^* \nu^*) = \frac{1}{|s^{-1}(s(e))|}\Phi(\gamma \nu^*).$$

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Theorem

Let E be a row-finite directed graph. Then the canonical conditional expectation $\Phi\colon C^*(E) \to C_0(E^0)$ is faithful.

Voiculescu's reduced amalgamated free product:

Given $\{(A_\iota, \Phi_\iota),\ \iota \in I\}$, unital C^*-algebras, $A_0 \subseteq A_\iota$, with conditional expectations $\Phi_\iota \colon A_\iota \to A_0$, the *reduced amalgamated free product* (A, Φ) is uniquely determined by:

1. A is a unital C^*-algebra and $\exists$ unital $*$-homomorphisms $\sigma_\iota \colon A_\iota \to A$ s.t. $(\sigma_\iota)_{|A_0} = (\sigma_{\iota'})_{|A_0}$ for all $\iota, \iota' \in I$.
2. A is generated by $\bigcup_{\iota \in I} \sigma_\iota(A_\iota)$.
3. $\Phi \colon A \to A_0$ is a conditional expectation such that $\Phi \circ \sigma_\iota = \Phi_\iota$ for all $\iota \in I$.
4. For $(\iota_1, \dots, \iota_n) \in \Lambda(I)$ and $a_j \in \ker \Phi_{\iota_j}$ we have $\Phi(\sigma_{\iota_1}(a_1) \cdots \sigma_{\iota_n}(a_n)) = 0$. Here, $\Lambda(I)$ denotes the family of indices $(\iota_1, \dots, \iota_n)$, $n \geq 1$, such that $\iota_i \neq \iota_{i+1}$ for $i = 1, \dots, n-1$.
5. If $c \in A$ is such that $\Phi(a^* c^* c a) = 0$ for all $a \in A$ then $c = 0$.

It is known that $\Phi\colon A \to A_0$ is faithful in case all the conditional expectations $\Phi_\iota\colon A_\iota \to A_0$ are faithful.

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Note that $C^*_{\mathrm{red}}(E, C)$ comes with a faithful conditional expectation $\Phi\colon C^*_{\mathrm{red}}(E, C) \to A_0 = C_0(E^0)$.

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Remark

There is also a definition of $C^*_{\mathrm{red}}(E, C)$ in case $|E^0| = \infty$.

Theorem

If E is a (non-separated) row-finite graph then the canonical map $C^*(E) \longrightarrow C^*_{\mathrm{red}}(E)$ is an isomorphism.

The examples revisited

Observe first that in case $|E^0| = 1$, we get the free product result $C^*(E, C) \cong *_{X\in C} \mathcal{O}_{|X|}$.

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$$C^*(E(m,n), C(m,n)) \cong M_{n+1}(\mathbb{C}) *_{\mathbb{C}^2} M_{m+1}(\mathbb{C}),$$

where $\mathbb{C}^2 \to M_{n+1}$ is given by sending $(1,0)$ to $e_{11} + \cdots + e_{nn}$ and $(0,1)$ to $e_{n+1,n+1}$, and similarly for $\mathbb{C}^2 \to M_{m+1}$.

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Combining this with the isomorphism

$$M_{n+1}(U^{\mathrm{nc}}_{m,n}) \cong C^*(E(m,n), C(m,n))$$

we get

$$M_{n+1}(U^{\mathrm{nc}}_{m,n}) \cong M_{n+1}(\mathbb{C}) *_{\mathbb{C}^2} M_{m+1}(\mathbb{C}).$$

The canonical conditional expectations Φ_X and Φ_Y are easily seen to correspond to the maps $\phi\colon M_{n+1}(\mathbb{C}) \to \mathbb{C}^2$ and $\psi\colon M_{m+1}(\mathbb{C}) \to \mathbb{C}^2$ given by

$$\phi([a_{ij}]) = (\frac{1}{n}\sum_{i=1}^{n} a_{ii},\ a_{n+1,n+1}),$$

$$\psi([b_{ij}]) = (\frac{1}{m}\sum_{j=1}^{m} b_{jj},\ b_{m+1,m+1}).$$

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The C^*-algebra $C^*_{\mathrm{red}}(E(m,n), C(m,n))$ is *simple*, but the C^*-algebra $C^*(E(m,n), C(m,n))$ is far from being simple.

nuclearity/exactness

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- The graph C^*-algebra $C^*(E)$ is nuclear for all row-finite graphs.
- For any separated graph (E, C), the reduced C^*-algebra $C^*_{\text{red}}(E, C)$ is exact.
- In general $C^*_{\text{red}}(E, C)$ is not nuclear, e.g.

$$C^*_{\text{red}}(E, C) = C^*_{\text{red}}(\mathbb{F}_2),$$

where (E, C) is the separated graph with one vertex v and two edges e_1, e_2, and $C = \{\{e_1\}, \{e_2\}\}$.

Open questions

Open Problems

1. When is the reduced graph C*-algebra of a finitely separated graph simple?
2. When is the full or reduced graph C*-algebra of a finitely separated graph finite?
3. Assume that $C^*_{\mathrm{red}}(E, C)$ is infinite. Is it then purely infinite? Does it at least have real rank zero?

Conjecture

Let (E, C) be a finitely separated graph. Let $M(E, C)$ be the abelian monoid with generators $\{a_v \mid v \in E^0\}$ and relations given by $a_v = \sum_{e \in X} a_{r(e)}$ for all $v \in E^0$ and all $X \in C_v$. Then the natural map

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is an isomorphism.

• Let $L(E, C)$ be the dense $*$-subalgebra of $C^*(E, C)$ generated by the canonical generators of $C^*(E, C)$. Then (Goodearl, A)

$$M(E, C) \cong \mathcal{V}(L(E, C)).$$

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- The answer is positive if we look at *stable* K-theory:

$$\mathrm{Grot}(M(E, C)) \cong K_0(C^*(E, C)).$$

Let

$$\langle a_1, \dots, a_n \mid \sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n} s_i a_i \rangle$$

be a presentation of the one-relator abelian monoid M, where $a_1, \dots, a_n$ are free generators, and $r_i + s_i > 0$ for all i. Let (E, C) be the finitely separated graph:

1. $E^0 := \{v, w_1, w_2, \dots, w_n\}$.
2. v is a source, and all the w_i are sinks.
3. For each $i \in \{1, \dots, n\}$, there are exactly $r_i + s_i$ edges with source v and range w_i.
4. $C = C_v = \{X, Y\}$, where X contains exactly s_i edges $v \to w_i$ for each i, and Y contains exactly r_i edges $v \to v_i$ for each i. Thus, $E^1 = X \sqcup Y$.

We call a separated graph constructed in this way a *one-relator separated graph*. As a particular example, we may consider the presentation $\langle a \mid ma = na \rangle$, with $1 \leq m \leq n$. This gives rise to the separated graph $(E(m,n), C(m,n))$.

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Remark

Observe that

$$M(E,C) \cong \langle a_1, \dots, a_n \mid \sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n} s_i a_i \rangle.$$

Indeed, given any abelian conical monoid S we can construct a separated graph (E,C) such that $S \cong M(E,C)$.

Theorem

Let (E, C) *be the one-relator separated graph associated to the presentation*

$$\langle a_1, \dots, a_n \mid \sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n} s_i a_i \rangle.$$

Set $M = \sum_{i=1}^{n} r_i$ *and* $N = \sum_{i=1}^{n} s_i$*, and assume that* $2 \leq M \leq N$*. Then* $C^*_{\text{red}}(E, C)$ *either is purely infinite simple or has a faithful tracial state, and it is purely infinite simple if and only if* $M < N$ *and there is* $i_0 \in \{1, \dots, n\}$ *such that* $s_{i_0} > 0$ *and* $r_{i_0} > 0$*. Moreover, if* $N + M \geq 5$ *and* $C^*_{\text{red}}(E, C)$ *is finite, then it is simple with a unique tracial state.*

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Set $M = \sum_{i=1}^{n} r_i$ and $N = \sum_{i=1}^{n} s_i$, and assume that $2 \leq M \leq N$. Then $C^*_{\text{red}}(E, C)$ either is purely infinite simple or has a faithful tracial state, and it is purely infinite simple if and only if $M < N$ and there is $i_0 \in \{1, \dots, n\}$ such that $s_{i_0} > 0$ and $r_{i_0} > 0$. Moreover, if $N + M \geq 5$ and $C^*_{\text{red}}(E, C)$ is finite, then it is simple with a unique tracial state.

Corollary

If $1 < m < n$ then $C^*_{\text{red}}(E(m, n), C(m, n))$ is purely infinite simple.

Lemma

Let F be the free abelian monoid on free generators $a_1, a_2, \dots, a_n$. Let

$$x = \sum_{i=1}^{n} r_i a_i, \qquad y = \sum_{i=1}^{n} s_i a_i$$

be nonzero elements in F. Let M be the conical abelian monoid $F/\sim$ where $\sim$ is the congruence on F generated by (x, y). Then M contains infinite elements if and only if either $x < y$ or $y < x$ in the usual order of F.

Recall that a C*-algebra A is termed *residually finite dimensional* if it admits a separating family of finite-dimensional $*$-representations.

Using

S. Armstrong, K. Dykema, R. Exel, H. Li, *On embeddings of full amalgamated free product C*-algebras*, Proc. Amer. Math. Soc. **132** (2004), 2019–2030,

we get

Proposition

Let (E, C) be the separated graph associated to the presentation $\langle a_1, \dots, a_n \mid \sum_{i=1}^n r_i a_i = \sum_{i=1}^n s_i a_i \rangle$. Consider the nonzero vectors $\mathbf{r} = (r_1, \dots, r_n)$ and $\mathbf{s} = (s_1, \dots, s_n)$ in $\mathbb{Z}^n$. Then the following conditions are equivalent:

(i) $C^*(E, C)$ is residually finite dimensional.

(ii) $C^*(E, C)$ admits a faithful tracial state.

(iii) $C^*(E, C)$ is stably finite.

(iv) $C^*(E, C)$ is finite.

(v) $\mathbf{r} \not< \mathbf{s}$ and $\mathbf{s} \not< \mathbf{r}$ in the usual order of $\mathbb{Z}^n$.

Example

There exists separated graphs (E, C) such that $M(E, C)$ is a stably finite monoid and $C^*(E, C)$ is a stably finite C*-algebra, but $C^*_{\mathrm{red}}(E, C)$ is purely infinite simple, and moreover the natural map $M(E, C) \to \mathcal{V}(C^*_{\mathrm{red}}(E, C))$ is not injective.

Proof.

Take for instance the separated graph associated to the one-relator monoid $\langle a, b \mid 3a + 2b = 2a + 4b \rangle$. By the Lemma, $M(E, C)$ is a stably finite monoid and, by the Proposition, $C^*(E, C)$ is a stably finite C*-algebra. However, by the Theorem, we have that $C^*_{\mathrm{red}}(E, C)$ is purely infinite simple. In particular we obtain that $\mathcal{V}(C^*_{\mathrm{red}}(E, C)) \setminus \{0\} = K_0(C^*_{\mathrm{red}}(E, C))$ is cancellative. Thus $a \neq 2b$ in $M(E, C)$ but $a = 2b$ in $\mathcal{V}(C^*_{\mathrm{red}}(E, C))$. This shows the result. □

Thank you very much for your attention!!!